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THE MOTION OF A CHARGED PARTICLE NEAR A ZERO FIELD POINT

Roger van Norton

July 15, 1961

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ABSTRACT

The behavior of charged particles in a plasma-free cusp field, or in a cusp field where there is very little plasma, is studied. Grad[1] and Berkowitz[2] have discussed particle losses from a cusped high β plasma with a sharp boundary. The numerical computations described in this report may provide a valid picture of the behavior of particles outside such a body of plasma or in the early stages of creation of such a cusped plasma in the low β limit. The computations also provide a striking illustration of orbits which neither have constant magnetic moment nor behave so irregularly as to defy explanation.

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THE MOTION OF A CHARGED PARTICLE NEAR A ZERO FIELD POINT

1. Introduction.

Orbits of a single charged particle in an imposed magnetic field were computed numerically on the IBM 704 at the IMS Atomic Energy Commission Applied Mathematics and Computing Center. The magnetic field is that which arises very near the zero field point of a counter Helmholtz field

$$\vec{B} = \text{const.}(x\hat{i} + y\hat{j} - 2z\hat{k}) \qquad (1.1)$$

i.e. the coils are at infinity. Figure 1 shows a portion of this field.

A catalogue of all possible orbits would be too long and too confusing to be of value. It is natural to wish to separate the orbits into a few broad classes defined in terms of the gross behavior of particles over a large but finite period of time. For example, it is known that a large class of orbits always remain so far from the zero-field point that the magnetic moment of such a particle is, for all practical purposes, a constant. Such a particle spirals about a guiding center field line and moves along that line (with some precession due to the non-uniformity of the field) in one direction until it is reflected in a region of convergent field lines. On the other hand, a particle passing very near the zero-field point executes a complicated orbit and undergoes large changes in its magnetic moment (See Figure 2). The calculations cited in the sequel are aimed at defining and providing a gross description of a class of orbits which are intermediate to these two extremes.

2. Equations of Motion and Auxiliary Definitions.

The equation of motion of a single charged particle in an imposed time-independent magnetic field is

$$m \frac{d^2\vec{X}}{dt^2} = c \left(\frac{d\vec{X}}{dt} \times \vec{B}\right) \tag{2.1}$$

Here $\vec{X}(t)$ is the position of the particle and $\vec{B}(\vec{X})$ is the magnetic field at that position. The constant depends on the charge and mass of the particle and on the units used.

Using the particular magnetic field (1.1), and choosing convenient units, the equation of motion (2.1) can be rewritten in components

$$\begin{aligned} \ddot{x} &= 2\dot{y}z + \dot{z}y \\ \ddot{y} &= -2\dot{x}z - \dot{z}x \\ \ddot{z} &= -\dot{x}y + \dot{y}x \end{aligned} \tag{2.2}$$

or

$$\ddot{\vec{X}} = - \dot{\vec{X}} \times \vec{B}.$$

It is clear that the speed

$$V = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} = \text{const.} \tag{2.3}$$

is a first integral of this system. Another constant of the motion is

$$-\dot{x}y + \dot{y}x + (x^2 + y^2)z = \text{const.} \tag{2.4}$$

more often written as

$$r^2\dot{\theta} + \dot{r}^2 z = \text{const.} \tag{2.5}$$

A combination of these two constants shows that an orbit is confined to the region

$$\left|\frac{k}{r} - rz\right| \leq V \tag{2.6}$$

where $k = r^2(0)(\dot{\theta}(0) + z(0))$. Figure 2 shows an orbit and the corresponding region defined in (2.6).

A list of definitions and nomenclature follows.

$r = (x^2 + y^2)^{1/2}$

$\Theta = \tan^{-1} y/x$

$V_\perp = \lvert\dot{\vec{X}} \times \vec{B}\rvert/B,$	particle speed perpendicular to the field
$V_{\parallel} = \dot{\vec{X}} \cdot \vec{B}/B,$	particle speed parallel to the field
$\mu = V_\perp^2/2B,$	magnetic moment of the particle
$\phi = \frac{r^2}{2} - z^2,$	the scalar potential of the magnetic field
$\psi = r^2 z,$	the stream function
$B = (r^2 + 4z^2)^{1/2}$	the magnitude of the field

Whenever $X(t)$ is a solution of (2.2), then $a\,X(at)$ is also a solution. Therefore it is necessary to consider only orbits for a single particle speed. This similarity principle will then yield orbits for all other speeds by appropriate choices of the parameter a.

A turning point of a particle is a point on the orbit where the path of the particle is perpendicular to the magnetic field. Equivalently, a turning point is a point of the orbit such that

$$\dot{\vec{X}} \cdot \nabla \phi = V \frac{d\phi}{ds} = 0 \tag{2.7}$$

where s is arc length measured along the particle's path.

3. Description of Results.

A typical particle orbit of the type considered has the general appearance shown in Figure 3. A particle is started at a turning point T_0, travels until it is reflected in the radial mirror at the turning point T_1, and returns to be reflected at the axial mirror at the turning point T_2. The orbits are very adiabatic near the turning points but are not at all adiabatic in the region of weak field.

A reasonable measure of the non-adiabaticity of the orbit is given by

$$\frac{\delta_2 \phi}{\phi} = [-\phi(T_2) + \phi(T_0)]/\phi(T_0) \tag{3.1}$$

As $z(T_0) \to \infty$ and $\psi(T_0)$ remains bounded, this ratio (when

of moderate size) is asymptotically equal to

$$2\,\frac{\delta z}{z} = -2[z(T_2) - z(T_0)]/z(T_0) \tag{3.2}$$

or

$$2\,\frac{\delta_2\mu}{\mu} = -2[\mu(T_2) - \mu(T_0)]/\mu(T_0). \tag{3.3}$$

For the orbits described, these approximate relations have only a small error.

In the computations the initial position T_0 of the particle is specified (see Fig. 4) as follows:

(i) A field line is given; this is the initial guiding center line.

(ii) A value of ϕ is specified; this distinguishes a point X_g on the guiding line.

(iii) The radius of gyration $\lambda_g = V/B_g$ is calculated. Here B_g is the field strength at X_g.

(iv) The initial position is taken to be a point on the circle with center at X_g, radius λ_g, and lying in the plane perpendicular to the guiding line.

Once initial position of the particle is known, then the initial velocity of the particle is chosen to be perpendicular to the radius vector of the aforementioned circle and so that its initial position is a turning point. Since the field is not uniform, the phase angle on the circle about X_g can not be neglected.

For each initial $\phi(T_0)$, a few initial guiding center lines separated by one radius of gyration were selected, and for each guiding center line many initial phase angles were selected in order to determine the non-adiabaticity of the orbits. Figure 5 shows $\frac{\delta_2 \phi}{\phi}$ as a function of phase. The constant $\frac{-v}{\phi(T_0)}$ of this figure differs very little from $\frac{2\lambda_g}{z_g}$. The quantity $\frac{\lambda_g}{z_g}$, which is computed at the orbits turning point near the axis, is simply the ratio of the radius of gyration and the distance of the turning point from the plane of symmetry. Each graph refers to a particular guiding centerline at the indicated value of $\phi(T_0)$. Each point on such a graph required a complete orbit calculation* from the turning point T_0 to the turning point T_1 and back to the final turning point T_2.

The left hand graph shows $\frac{\delta_2 \phi}{\phi}$ as a function of phase. Some orbits were followed through a second reflection in the radial mirror and a second return to a turning point in the axial mirror. Their relative change in ϕ, which is called $\frac{\delta_4 \phi}{\phi}$, is plotted to the right. Notice that the abscissa scale is expanded by a factor of 64.

If the detailed particle orbits are examined for the cases represented in these graphs, it is seen that the particle remains remarkably faithful to its original guiding center line even though its magnetic moment may have changed by 30°/o or more.

Thus three classes of orbits may be distinguished:

---

* Each complete orbit computation involved from 4 minutes to 40 minutes of 704-time.

(i) A large class which avoids weak field regions. The particles in such orbits move from turning point to turning point with a sensibly constant magnetic moment. Such an orbit would look much like the orbit in Figure 3 but T_2 would be nearer to T_0.

(ii) A small transition class of orbits which might be conveniently defined as those for which

$$0.1 \leq \max\left|\frac{\delta_2 \emptyset}{\emptyset}\right| \leq 0.3$$

where the maximum is over all phase angles. An example is shown in Figure 5. For a constant $\emptyset(T_0)$, the guiding lines for these transition class orbits lie in a band only about 3 or 4 radii-of-gyration wide at T_0 and at a distance of 16 to 18 radii of gyration from the axis for the example given in figure 5.

(iii) A set of orbits, whose initial turning points are even nearer the axis, for which the adiabatic invariant is not valid. The epitome of this class is the subclass of orbits which initially tend to encircle the axis of the system. Figure 2 shows such an orbit. Starting at T_0 the orbit initially encircles the axis several times. Since the orbit is plotted in cylindrical coordinates, it appears to be traveling roughly parallel to the axis rather than revolving about the axis.

Figure 6 shows this rough separation of orbits into the three classes. The region in which the orbit's initial turning

point lies determines the orbit's class. (Notice that ordinate has been magnified by a factor of ten in order to clarify the diagram.)

4. Conclusions.

From the computations for the transition class of orbits, several conclusions may be drawn.

(i) The class of orbits for which

$$0.1 \leq \max\left|\frac{\delta_2 \phi}{\phi}\right| \leq 0.3, \quad \text{where the maximum is over all phase angles,}$$

is small. The distance from the axis of initial turning points for such orbits is many times greater than the thickness of the band they occupy.

(ii) An examination of the details of moderately non-adiabatic orbits shows that such an orbit may be considered to consist of several fairly distinct segments.

(a) First, there is a segment of the orbit from the initial turning to a region where the field is smallest for the orbit. This part of the orbit is very nearly a guiding center motion and the magnetic moment is nearly constant.

(b) Second, there is a small part of the orbit in a region of weak field where the magnetic moment undergoes large oscillations.

(c) The orbit continues into a region of stronger field. The motion is again very nearly a guiding center motion but with a magnetic moment different from its original value.

(d) The particle is reflected at T_1, a turning point in the radial mirror. Consider orbits with the same $\phi(T_0)$, the initial scalar potential, but with different phase angles at T_0; all such orbits have essentially the same initial magnetic moments. When the orbits cross the region of weak magnetic field, their magnetic moments suffer different changes from their original values.

These changes depend on the initial phase angles. Therefore, such particles will advance different distances into the radial mirror. The detail of the turning point in Figure 3 shows that at the turning point the phase is a very rapidly changing function of distance along the guiding center line. Thus, any difference in magnetic moments results in a large phase difference after the particles are turned back. The phase difference is so rapidly varying a function of the magnetic moment difference that it may be conveniently considered as a mechanism for randomizing the phases of the particles.

(e) Finally, the particle travels back towards the turning point T_2. Its motion is as described in (c), (b) and (a) above.

When $\frac{\delta_2 \phi}{\phi}$ is computed as a function of the initial phase, the combination of the three following effects is observed.

(a) A change in magnetic moment which occurs at the first traversal of the weak field region.

(b) Randomization of phase at the turning point T_1.

(c) Another change in magnetic moment in the return traversal of the weak field region.

(iii) The variation of $\frac{\delta_2 \phi}{\phi}$ as a function of the initial phase angle is quite rapid; this is caused by phase randomization at the second turning point T_1. This rapid variation and the narrowness of the transition zone combine to support Grad's suggestion that particles which pass near the zero field point may be treated statistically.

(iv) The position of the transition band has been found for a few values of the scalar potential. Further computations have given an empirical determination of its asymptotic behavior as the initial turning point moves farther from the origin. Let r_c, z_c denote the position band. The distance of the transition band from the axis, r_c, decreases to zero as z_c tends to infinity, but $\frac{r_c}{\lambda_c}$, the same distance but measured in units of radii of gyration, tends to infinity as z_c tends to infinity. Furthermore, the transition band moves out to more distant field lines as z_c increases. These three items of description may be rewritten

$$r_c \sim z_c^{-.43}$$

$$\frac{r_c}{\lambda_c} \sim z_c^{.57}$$

$$\psi_c \sim z_c^{.14}$$

These empirical formulas provide a description of the behavior of the transition zone as a function of $\phi(T_0)$, or equivalently as a function of the particle's speed for a constant $\phi(T_0)$.

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[2] Berkowitz, J., Theory of Cusped Geometries, II. Particle Losses, NYO-2536, Inst. Math. Sci., N.Y.U., January 6, 1959.

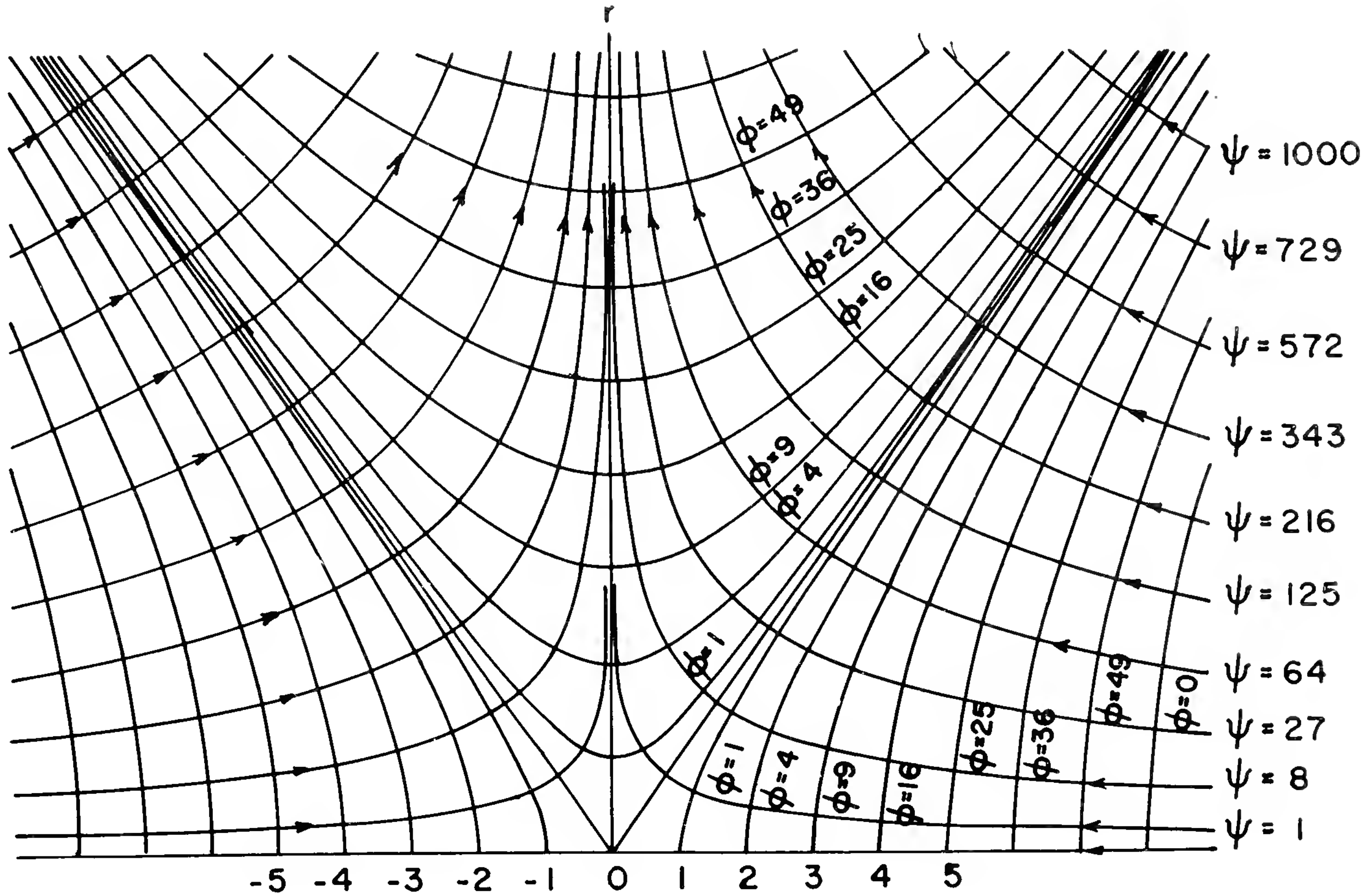


Figure 1: The stream function ψ is constant on magnetic field lines. The equipotential surfaces (ϕ = const.) are orthogonal to the magnetic field lines.

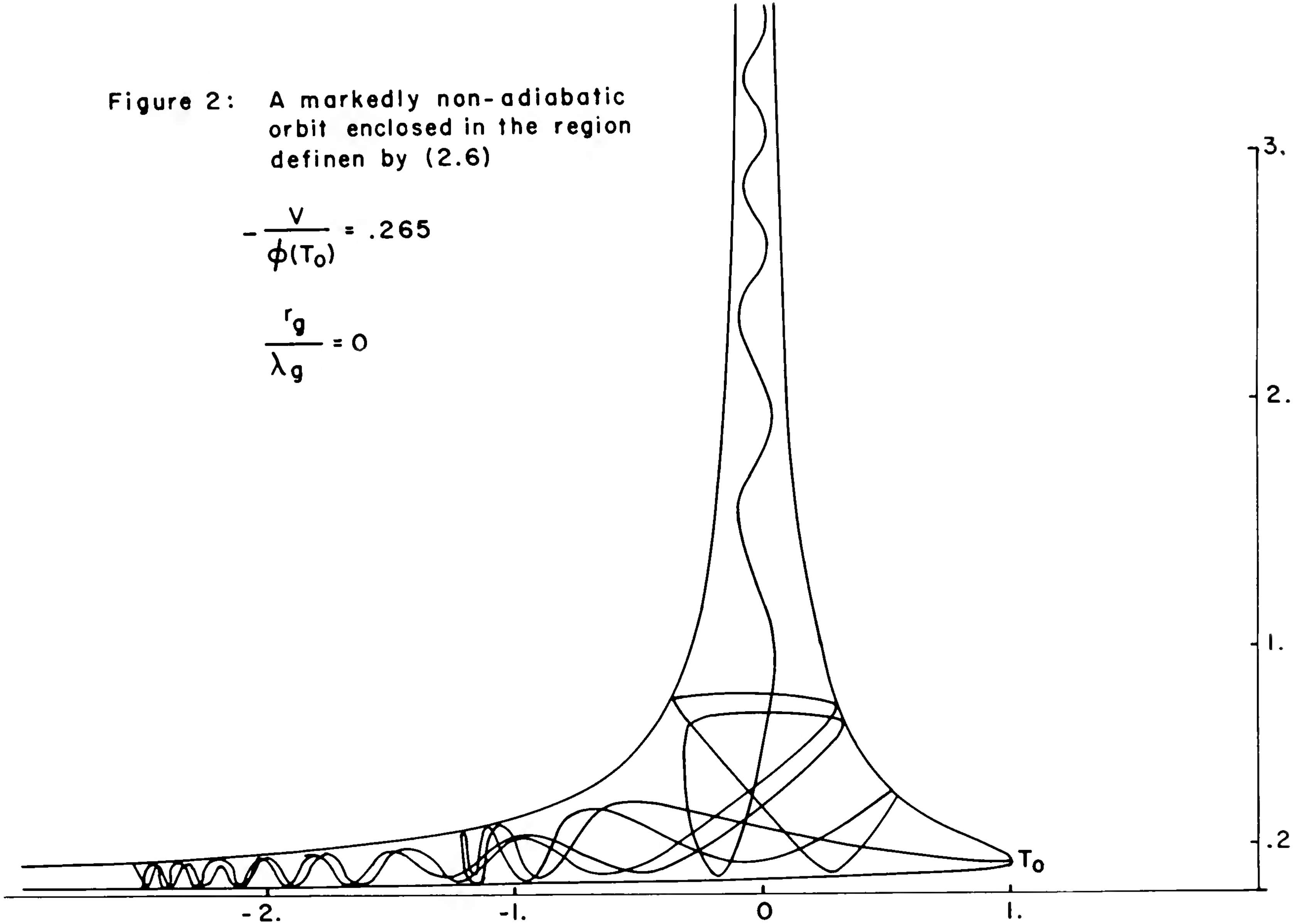


Figure 2: A markedly non-adiabatic orbit enclosed in the region definen by (2.6)

$$-\frac{V}{\phi(T_o)} = .265$$

$$\frac{r_g}{\lambda_g} = 0$$

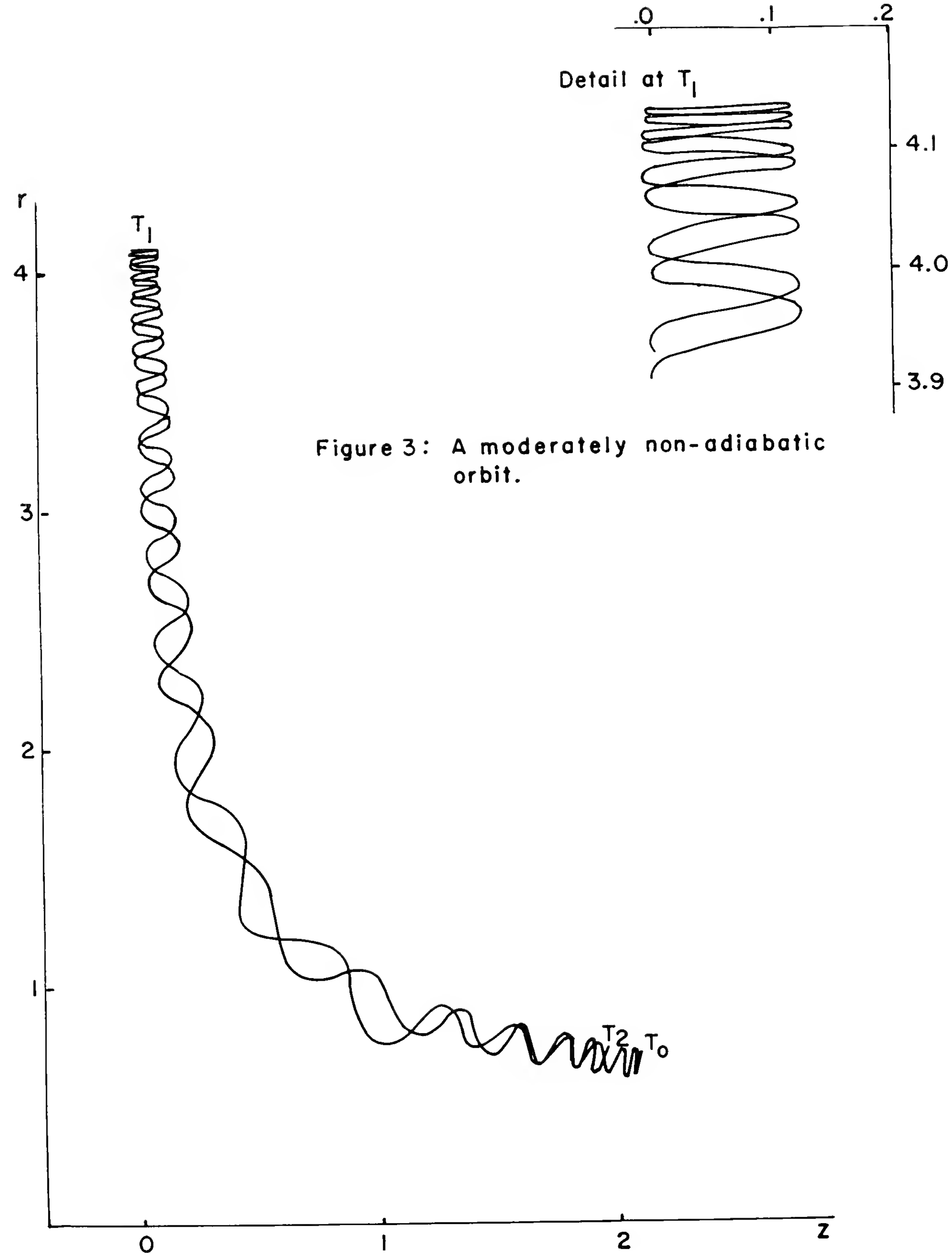


Figure 3: A moderately non-adiabatic orbit.

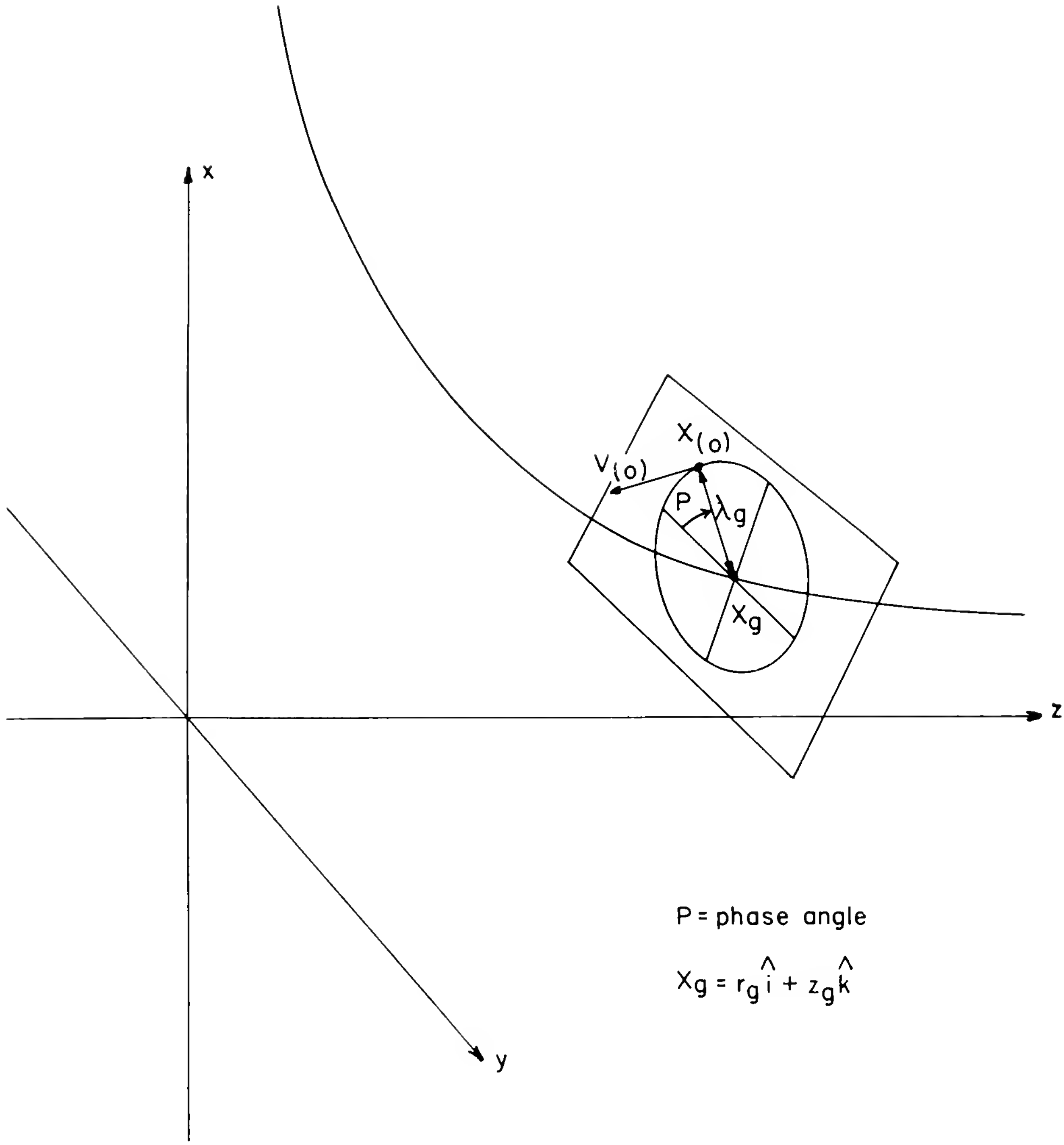
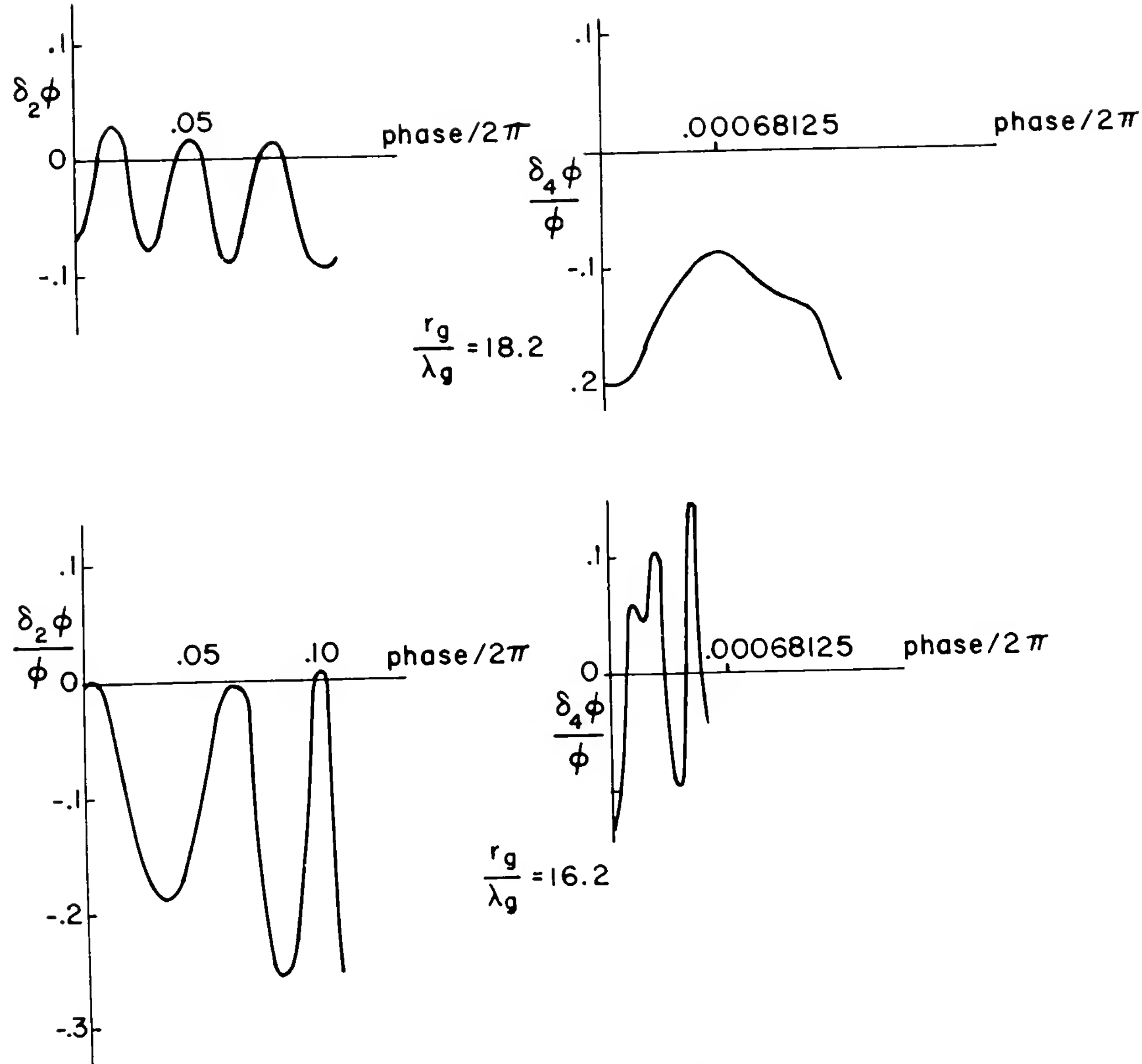


Figure 4: The initial position and velocity of a particle.



$-v/\phi(T_0) = 1.65 \times 10^2$

Figure 5

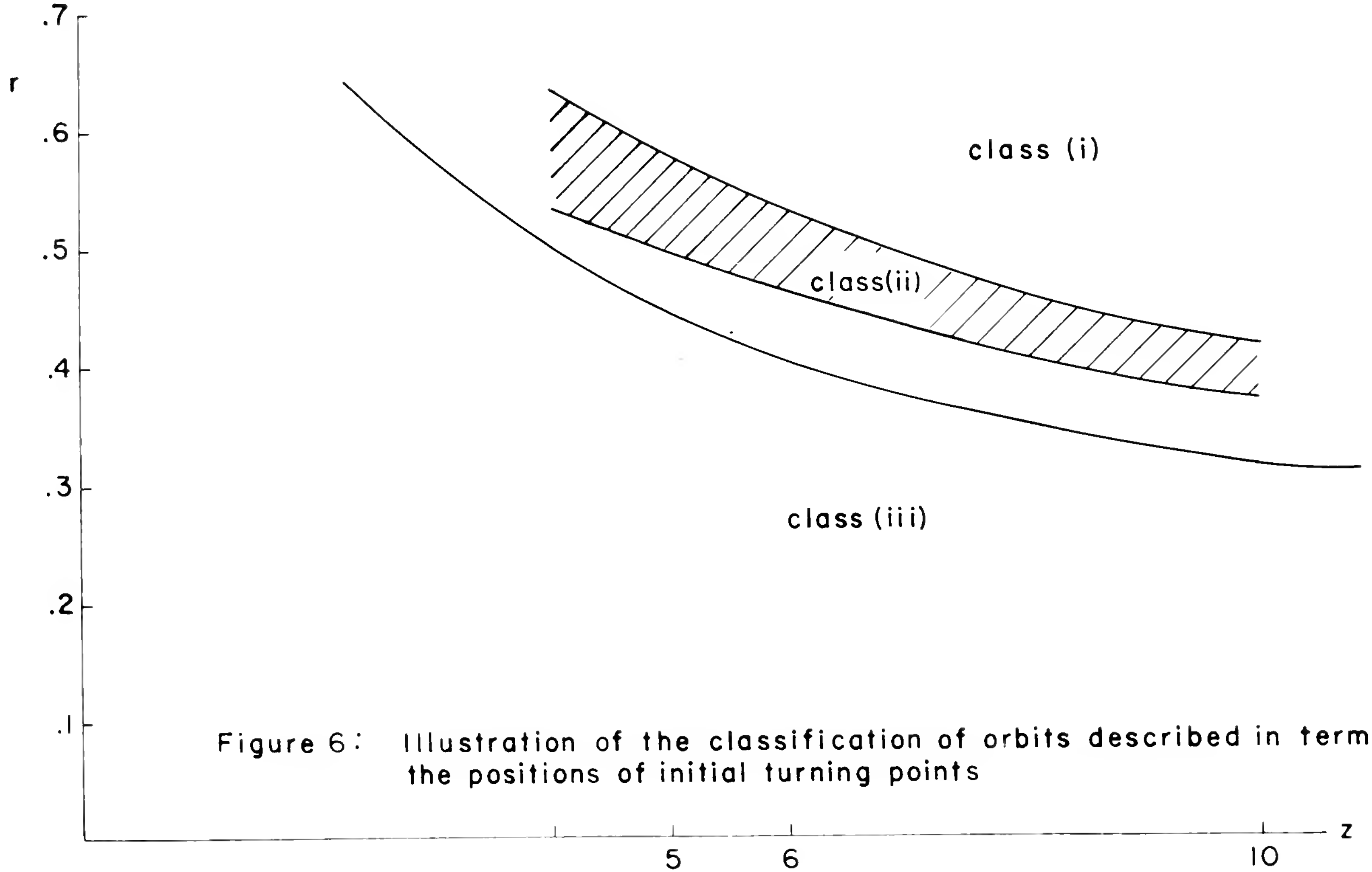


Figure 6: Illustration of the classification of orbits described in terms of the positions of initial turning points

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